NONPERTURBATIVE FORMULAS FOR CENTRAL FUNCTIONS OF SUPERSYMMETRIC GAUGE THEORIES

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Abstract

For quantum field theories that flow between ultraviolet and infrared fixed points, central functions, defined from two-point correlators of the stress tensor and conserved currents, interpolate between central charges of the UV and IR critical theories. We develop techniques that allow one to calculate the flows of the central charges and that of the Euler trace anomaly coefficient in a general N=1 supersymmetric gauge theory. Exact, explicit formulas for $SU(N_c)$ gauge theories in the conformal window are given and analysed. The Euler anomaly coefficient always satisfies the inequality $a_{UV} - a_{IR} > 0$. This is new evidence in strongly coupled theories that this quantity satisfies a four-dimensional analogue of the c-theorem, supporting the idea of irreversibility of the RG flow. Various other implications are discussed.

1 Introduction

In two dimensions there are many known examples of quantum field theories that flow under the renormalization group (RG) from a conformal fixed point in the ultraviolet to another fixed point in the infrared. Often, it is possible to work out exact results, using properties and techniques that are special to two dimensions. The operator product expansions of two stress tensors or two conserved currents contain central charges c and k which encode fundamental properties of the conformal theories that appear at the limits of the flow. The Zamolodchikov c-theorem [1] states the important inequalities $c_{UV} - c_{IR} > 0$ and $k_{UV} - k_{IR} > 0$ that place constraints on the flow and have a useful physical interpretation in terms of RG irreversibility and the thinning out of degrees of freedom as one moves to longer distances.

In four dimensions it is more difficult to establish the existence of conformal fixed points. When they exist, a quantum field theory can be described as a radiative interpolation between pairs of four-dimensional conformal field theories. The problem is then to identify relevant physical quantities and study their renormalization group flow from one fixed point to the other. We call this problem the RG interpolation. While the RG interpolation seems to be very difficult in the general case, it simplifies considerably for supersymmetric theories, where many examples of interacting conformal fixed points have been studied. In particular, supersymmetric gauge theories in the "conformal window" [2] have nontrivial IR fixed points; this is also the subset of theories with electric-magnetic duality. The general relation between the trace anomaly and the chiral anomaly of the R-current in supersymmetry allows us to solve the RG interpolation problem for supersymmetric gauge theories, and the principal application of these techniques is to theories in the conformal window.

Another difficulty of four dimensions is that no analogue of the c-theorem has been proved, although the coefficient a of the Euler density in the curved space trace anomaly has been proposed [3] as a c-theorem candidate, and the flow $a_{UV} - a_{IR}$ is known to be positive in all cases where it can be tested. We shall have more to say about this later.

In two-dimensional conformal theories the operator product expansions of two stress tensors $T_{\mu\nu}$ or conserved currents J_μ are closed, namely no new operators appear, but the situation is more complicated in higher dimensions. As in two dimensions, the TT and JJ OPE's define primary central charges c and b, respectively. But these OPE's are not closed [4, 5]. New operators Σ with anomalous dimension appear, and the OPE's of the new operators define secondary central charges c' and b' [5]. A Σ operator can be used to deform the theory off-criticality and its anomalous dimension h then coincides with the critical value of the slope of the β-function [5]. This property indicates that information about the off-critical theory can be obtained by studying its critical limit.

It is reasonable to suggest that the central charges c, c', b, b', the Euler coefficient a and the anomalous dimension h are the fundamental parameters of four-dimensional conformal theories and to study these quantities, which depend on the dimensionless couplings of the theory. The lowest order, two-loop radiative corrections to the central charge c were obtained by several authors and in particular by Jack [6] in the most general renormalizable theory of

scalar, spinor, and gauge fields. In [7] the secondary central charge c' was computed to two-loop order for general $N = 1$ SUSY gauge theories and Jack's result for c was specialized to the case of supersymmetric couplings. The general $N = 1$ theory contains the extended $N = 4$ theory and the $N = 2$ theory with the critical number of hypermultiplets as special cases, and it was observed that c and c' are constant on the well-known marginal fixed lines of those theories. It is a consequence of the c-theorem in two dimensions that the central charge c is constant along lines of marginal deformation. This suggests that the central charges are invariants of superconformal field theory in four dimensions (SCFT$_4$) - they do not change within families of continuously connected theories. The anomalous dimension h does vary along marginal lines.

In ref. [8] non-perturbative definitions of primary and secondary central functions were given in terms of two- and four-point correlators of the stress tensor and conserved currents. The central functions interpolate between the critical values of the central charges. One goal of the RG interpolation is to work out nonperturbative expressions for these functions. We are going to show that this can be achieved in supersymmetric theories for all primary central functions, as well as a special subclass of secondary central functions.

In particular, the major results of our analysis are non-perturbative formulas for the flow of the primary central functions b, c and a, that agree with perturbative calculations and can be applied within the conformal window of $SU(N_c)$ theories with N_f flavors to give exact formulas for the total flows $b_{UV} - b_{IR}$, $c_{UV} - c_{IR}$ and $a_{UV} - a_{IR}$ in terms of N_c and N_f and no other parameters. The first of these is strictly negative so that no b-theorem holds, while the second changes sign from positive to negative as N_f increases from $3N_c/2$ to $3N_c$. On the other hand, the flow of a is positive in the entire conformal window. A byproduct of our general analysis is the marginal constancy of the primary central functions to all orders in perturbation theory.

At the perturbative level, while the two-loop contribution to the primary flavor central charge b turns out to vanish along marginal lines, in agreement with the nonperturbative formulas, the radiative corrections to the secondary flavor central charge b' are not marginally constant. At the moment, we do not have a general nonperturbative treatment of secondary central functions. We hope to discuss our two-loop calculations, which are based on an interesting application of conformal symmetry to calculations of Feynman diagrams, and the situation of secondary central charges, in a subsequent paper [9].

The flow of the flavor central charge b is analysed by two methods, one in Section 2 and the other in Section 3. The second method is more direct, but only the first can be easily extended to the gravitational central charges c and a as described in Section 4. In Section 5, our formulae for the total flows $b_{UV} - b_{IR}$, $c_{UV} - c_{IR}$ and $a_{UV} - a_{IR}$ are discussed, and we give there our conclusions and outlook. In Appendix A, the component gravitational anomalies are obtained from their curved superspace counterparts.

2 Flow of the flavor central charge

We consider $SU(N_c)$ supersymmetric QCD with N_f quark flavors. The theory contains gauge superfields $V^a(x, \theta, \bar{\theta}), a = 1, \ldots, N_c^2 - 1$, whose physical components are the gauge potentials $A_\mu^a(x)$ and Majorana gauginos $\lambda^a(x)$. There are also chiral quark and anti-quark superfields, $Q^{\alpha i}(x, \theta)$ and $\widetilde{Q}_{\alpha i}(x, \theta)$, respectively, where $\alpha = 1, \ldots, N_c$ and $i = 1, \ldots, N_f$. The matter components of $Q^{\alpha i}$ are the complex scalars $\phi^{\alpha i}$ and Majorana spinors $\psi^{\alpha i}$, while $\widetilde{Q}_{\alpha i}$ contains $\tilde{\phi}_{\alpha i}$ and $\tilde{\psi}_{\alpha i}$. This electric theory has the usual gauge interactions and no superpotential. Later we will extend the treatment to the magnetic theory where there are additional gauge neutral chiral superfields and a cubic superpotential.

The theory has an anomaly-free $SU(N_f)_Q \times SU(N_f)_{\widetilde{Q}} \times U(1)_B$ global symmetry group, and there are conserved currents that appear as the $\theta\bar{\theta}$ components of the superfields $\bar{Q}t^A Q$, $\widetilde{Q}\tilde{t}^A\widetilde{Q}$, and $B = (Q\bar{Q} - \widetilde{Q}\widetilde{\bar{Q}})/N_c$. Here t^A and $\tilde{t}^A$ are matrix generators of the fundamental and anti-fundamental representations of $SU(N_f)$, respectively. The component Noether currents of the Euclidean signature theory are (see [7] for details of the notation)

$$J_\mu^A = -\bar{\psi}\gamma_\mu L t^A \psi + \bar{\phi} \overleftrightarrow{D}_\mu t^A \phi, \tag{2.1}$$

$$\widetilde{J}_\mu^A = -\bar{\tilde{\psi}}\gamma_\mu L \tilde{t}^A \tilde{\psi} + \bar{\tilde{\phi}} \overleftrightarrow{D}_\mu \tilde{t}^A \tilde{\phi},$$

$$J_\mu = \frac{1}{N_c}\left[\frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi - \frac{1}{2}\bar{\tilde{\psi}}\gamma_\mu\gamma_5\tilde{\psi} + \bar{\phi} \overleftrightarrow{D}_\mu \phi - \bar{\tilde{\phi}} \overleftrightarrow{D}_\mu \tilde{\phi}\right].$$

There are also classically conserved but anomalous Konishi and R currents that are given by

$$K_\mu = \left[\frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi + \frac{1}{2}\bar{\tilde{\psi}}\gamma_\mu\gamma_5\tilde{\psi} + \bar{\phi} \overleftrightarrow{D}_\mu \phi + \bar{\tilde{\phi}} \overleftrightarrow{D}_\mu \tilde{\phi}\right], \tag{2.2}$$

$$R_\mu = \frac{1}{2}\bar{\lambda}^a\gamma_\mu\gamma_5\lambda^a - \frac{1}{6}(\bar{\psi}\gamma_\mu\gamma_5\psi + \bar{\tilde{\psi}}\gamma_\mu\gamma_5\tilde{\psi}) + \frac{2}{3}(\bar{\phi} \overleftrightarrow{D}_\mu \phi + \bar{\tilde{\phi}} \overleftrightarrow{D}_\mu \tilde{\phi}).$$

The first is the $\theta\bar{\theta}$ component of $K = Q\bar{Q} + \widetilde{Q}\widetilde{\bar{Q}}$, while the second is the lowest component of the supercurrent superfield $J_{\alpha\dot{\alpha}}$ that also contains the stress tensor and supersymmetry currents. It is well known that K_μ and R_μ have internal anomalies [1], expressed by the operator equations (see [10])

$$\partial_\mu R^\mu = \frac{1}{48\pi^2}[3N_c - N_f(1 - \gamma)]F_{\mu\nu}^a\tilde{F}_{\mu\nu}^a, \qquad \partial_\mu K^\mu = \frac{N_f}{16\pi^2}F_{\mu\nu}^a\tilde{F}_{\mu\nu}^a. \tag{2.3}$$

The anomaly-free, RG invariant, combination [10] of K_μ and R_μ, namely

$$S_\mu = R_\mu + \frac{1}{3}\left(1 - \frac{3N_c}{N_f} - \gamma\right)K_\mu, \tag{2.4}$$

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[1]We distinguish between internal anomalies, involving the quantum gauge field, and external anomalies, involving an external classical gauge field.

will be important for us. The coefficient of K_μ is the numerator of the exact NSVZ [11] β-function

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f(1 - \gamma(g))}{1 - g^2 N_c/8\pi^2} \tag{2.5}$$

and $\gamma/2$ is the anomalous dimension of the superfield Q (or $\widetilde{Q}$).

We now let $J_\mu(x)$ denote any one of the conserved flavor currents of (2.1). Its two-point current correlation function has the form

$$\langle J_\mu(x) J_\nu(0) \rangle = \frac{1}{16\pi^4}(\partial_\mu \partial_\nu - \Box\, \delta_{\mu\nu}) \left(\frac{b(g(1/x))}{x^4} \right), \tag{2.6}$$

Since $J_\mu(x)$ is conserved and thus has no anomalous dimension, the Callan- Symanzik equation requires that $b(g(1/x))$ depend only on the running coupling $g(1/x)$ that satisfies

$$\mu\frac{\mathrm{d}g(\mu)}{\mathrm{d}\mu} = \beta(g(\mu)). \tag{2.7}$$

As discussed in [8] $b(g(1/x))$ is a primary central function that interpolates between flavor central charges at the fixed points g_{UV} and g_{IR} of the renormalization group flow. Specifically

$$b_{UV} = \lim_{x \longrightarrow 0} b(g(1/x)) = b(g_{UV}) \tag{2.8}$$
$$b_{IR} = \lim_{x \longrightarrow \infty} b(g(1/x)) = b(g_{IR})$$

The principal result of this section will be an exact non-perturbative formula for $b_{UV} - b_{IR}$. This will take some discussion, but the basic ideas are simple and we will list them here before we begin the derivation:

1. In the presence of an external source B_μ for the current J_μ, the trace anomaly of the theory includes the familiar internal anomaly plus an external anomaly of similar form,

$$\Theta = -\frac{3N_c - N_f(1 - \gamma)}{32\pi^2}\,(F_{\mu\nu}^a)^2 + \frac{1}{4}q(B_{\mu\nu})^2 \tag{2.9}$$

   where $F_{\mu\nu}^a$ and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are internal and external field strengths, respectively. The general form of (2.9) follows uniquely from gauge invariance, locality dimensional analysis, and parity [13]. As discussed in refs. [12, 10], the coefficient of $(F_{\mu\nu}^a)^2$ is the numerator of the β-function (2.5). The coefficient of $(B_{\mu\nu})^2$ is the subject of our investigation. Following the discussion of [8], that we review below, q can be identified [2] with the coefficient of a local term that appears in the explicit scale derivative of the regulated correlator (2.6). This coefficient is a scale dependent function $\widetilde{b}(g(\mu))$ that can be shown to have the same UV and IR limits as $b(g(\mu))$.

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[2]As discussed in Section 3, q can be actually identified with the limit as $e \to 0$ of the $U(1)$ β-function $\beta_e(g, e)$ of an extended $SU(N_f) \times U(1)$ gauge theory with $U(1)$ coupling constant e, and this provides an alternative method to derive the flow. We do not use this observation here, because only the present method can be easily extended to the gravitational central charges as discussed in Section 4.

2. The introduction of external sources can be done supersymmetrically by embedding $B_\mu(x)$ in an external gauge superfield $B(x, \theta, \bar{\theta})$ coupled to the superspace flavor current. The usual relation between the trace and $\partial_\mu R_\mu$ anomalies implies that

$$\partial_\mu R_\mu = \frac{3N_c - N_f(1 - \gamma)}{48\pi^2} \ F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu} - \frac{1}{6} q B_{\mu\nu} \widetilde{B}_{\mu\nu} \tag{2.10}$$

In this way the flow of the central function $b(g(\mu))$ is related to the flow of the anomalous correlator $\langle R_\mu(x) J_\nu(y) J_\rho(z) \rangle = \frac{\partial}{\partial B_\nu(y)} \frac{\partial}{\partial B_\rho(z)} \langle R_\mu(x) \rangle$.

3. The final ingredient is 't Hooft anomaly matching [14] for the internal anomaly-free current S_μ in (2.4). Since S_μ is quantum-conserved in the absence of sources, its external anomaly is scale-independent. This implies that the particular combination of the external ∂R and ∂K anomalies in (2.4) is independent of scale, and a useful non-perturbative expression for the flow of $\widetilde{b}(g(\mu))$ emerges from this observation.

To begin discussion of the first, and probably least familiar, ingredient, we consider the structure of the correlator (2.6) to any finite order of perturbation theory and construct a line of argument from which we extract an all-order result. We consider the computation of (2.6) in two stages. In the first stage we work at separated points and regulate all sub-divergences at the scale μ. To any finite order, $b(g(1/x))$ can be expressed in the form

$$b(g(1/x)) = \sum_{n \geq 0} b_n(g(\mu)) t^n, \qquad t = \ln(x\mu), \tag{2.11}$$

where $b_n(g)$ is a polynomial in g. Evaluating (2.11) at $x = 1/\mu$, we see that $b(g) = b_0(g)$. The Callan-Symanzik equations imply

$$\beta(g) b'_n(g) + (n + 1) b_{n+1}(g) = 0 \tag{2.12}$$

so that all $b_n(g)$ for $n \geq 1$ are proportional to $\beta(g)$ and can be expressed in terms of $b_0(g)$, $\beta(g)$ and derivatives.

When (2.11) is inserted in (2.6) one finds t^n/x^4 terms which are too singular to have a finite Fourier transform. To correct this we regulate this overall divergence at $x = 0$ using the generalized differential identity [8]

$$\frac{(\ln x\mu)^n}{x^4} = -\frac{n!}{2^{n+1}} \Box \sum_{k=0}^{n} \frac{2^k t^{k+1}}{(k+1)! \ x^2} - a_n \delta(x), \tag{2.13}$$

which is an application of the method of differential renormalization [15]. Any other method in which the overall divergence can be separated from sub-divergences would also work. For example, dimensional regularization in x-space might be used. The right hand side of (2.13) contains the Lorentz invariant solution of the differential equation $\Box f(x) = t^n/x^4$ which is unique up to the additive numerical constant a_n. Combining (2.11), (2.13) and (2.6), we find the fully regulated correlator

$$\frac{b(g(1/x))}{x^4} = -\sum_{n} b_n(g(\mu)) \left[\frac{n!}{2^{n+1}} \Box \sum_{k=0}^{n} \frac{2^k t^{k+1}}{(k+1)! \ x^2} + a_n \delta(x) \right]. \tag{2.14}$$

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The explicit scale derivative of (2.6) gives [16] the correlator $\langle J_\mu(x)J_\nu(0)\int d^4z\Theta(z)\rangle$ where Θ is the trace anomaly (2.9). We compute this scale derivative as $\mu\,\partial/\partial\mu$ acting on (2.14). The result is expressed as the sum of the local contribution of the $k=0$ term plus a non-local term proportional to $\beta(g)$ because of (2.12)

$$\mu\frac{\partial}{\partial\mu}\frac{b(g(1/x))}{x^4} = 2\pi^2\widetilde{b}(g(\mu))\delta(x) + \beta(g(\mu))\Box\frac{F(x)}{x^2}\ ,\tag{2.15}$$

where

$$\widetilde{b}(g(\mu)) = \sum_n b_n(g(\mu))\frac{n\,!}{2^n}\ .\tag{2.16}$$

$F(x)$ contains the sum of all $k \geq 1$ terms in the scale derivative, and the a_n have dropped out because they have vanishing scale derivative. The decomposition between local and nonlocal contributions in the above expression is not universal. For example, one can add an arbitrary x -independent function $A(g)$ to $F(x)$ and redefine $\widetilde{b}(g)$ as $\widetilde{b}(g) - \beta(g)A(g)$ in the local part. However, such contributions are proportional to $\beta(g(\mu))$ and vanish at the fixed points, so the total flow of the central function $b(g(1/x))$ can be computed from the flow of the function $\widetilde{b}$ as defined in (2.16).

It is easy to derive, using (2.12), a differential equation for $\widetilde{b}(g)$, namely

$$\beta(g)\frac{\partial\widetilde{b}(g)}{\partial g} + 2\widetilde{b}(g) = 2b(g),\tag{2.17}$$

which is solved by

$$\tilde{b}(g(\mu)) = \frac{1}{\mu^2}\int_0^{\mu^2} d\mu'^2 b(g(\mu')).\tag{2.18}$$

These formulas have been derived within perturbation theory, but we shall regard them as non-perturbative results. Either formula shows that the functions $b(g(\mu))$ and $\widetilde{b}(g(\mu))$ coincide at fixed points of the RG flow, as well as to the second loop order in perturbation theory around the free fixed point [3].

Finally, one can identify $\tilde{b}(g(\mu))$ with the coefficient q appearing in eq. (2.9). To show this we write the generating functional for the current correlation functions as the schematic path integral

$$e^{-\Gamma[B_\mu]} = \int[d\Phi]e^{-S[\Phi]+i\int d^4xJ_\mu(x)B_\mu(x)}.\tag{2.19}$$

The source couples just like an abelian gauge field without kinetic term. The scale derivative $\mu\frac{\partial}{\partial\mu}$ corresponds to the insertion of $\int d^4z\Theta(z)$ inside the path integral, so that

$$\mu\frac{\partial}{\partial\mu}e^{-\Gamma} = \int[d\Phi]e^{-S[\Phi]+i\int d^4xJ_\mu(x)B_\mu(x)}\int d^4z\left[-\frac{3N_c-N_f(1-\gamma)}{32\pi^2}\,(F^a_{\mu\nu})^2 + \frac{1}{4}q(B_{\mu\nu})^2\right]\tag{2.20}$$

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[3]To two-loop order there are no $\ln x\mu$ terms in (2.11) so that $\tilde{b}(g) = b(g)$.

with similar internal and external contributions. From (2.20) we see that the scale derivative of the current correlator satisfies

$$\mu\frac{\partial}{\partial\mu} \ \langle \ J_\mu(x)J_\nu(0)\rangle = \langle J_\mu(x)J_\nu(0)\int d^4z\Theta(z)\rangle \tag{2.21}$$

$$= q(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu})\delta^4(x) - \frac{3N_c - N_f(1-\gamma)}{32\pi^2} \ \langle J_\mu(x)J_\nu(0)\int d^4z(F^a_{\mu\nu})^2\rangle.$$

It has the same form as (2.15), namely the sum of a local term plus a non-local term proportional to $\beta(g(\mu))$, so that we can identify the coefficients of the local terms, again up to contributions $O(\beta(g(\mu))$ that vanish at fixed points. (Note that the last correlator in (2.21) is $O(g^4(\mu))$ and vanishes in the ultraviolet.) We can thus write

$$q = \frac{1}{8\pi^2}\widetilde{b}(g(\mu)), \tag{2.22}$$

with the understanding that possible $O(\beta(g(\mu)))$ corrections, that are irrelevant for the total flow of $\widetilde{b}(g(\mu))$, are omitted.

The next step is to use the $\partial_\mu R^\mu$ anomaly (2.10) which is the supersymmetric partner of the trace anomaly (2.9) to compute the flow of $\widetilde{b}(g(\mu))$. We need the anomalous correlation functions $\langle S_\mu(x)J_\nu(y)J_\rho(z)\rangle$, $\langle R_\mu(x)J_\nu(y)J_\rho(z)\rangle$ and $\langle K_\mu(x)J_\nu(y)J_\rho(z)\rangle$ and their flows. The anomalous divergence of such current correlators is one-loop exact only if all currents have no internal anomalies. Otherwise the so-called "rescattering graphs," which contain an internal triangle in which one of the currents communicates to a pair of gluons, are responsible for higher order non-local corrections [17]. Therefore 't Hooft anomaly matching holds for the correlator $\langle SJJ\rangle$ where S_μ is the anomaly-free current of (2.4), but not for $\langle RJJ\rangle$, and in general not for $\langle KJJ\rangle$. The anomalous Ward identities of these correlators can be written as the following equations for matrix elements of S_μ, R_μ and K_μ in the presence of the current source B_μ

$$\langle\partial_\mu S_\mu\rangle \equiv \frac{1}{48\pi^2}s_0 B_{\mu\nu}\widetilde{B}_{\mu\nu}, \tag{2.23}$$

$$\langle\partial_\mu R_\mu\rangle = -\frac{1}{48\pi^2}\widetilde{b}(g(\mu))B_{\mu\nu}\widetilde{B}_{\mu\nu} + \cdots,$$

$$\langle\partial_\mu K_\mu\rangle = -\frac{1}{16\pi^2}\tilde{k}(g(\mu))B_{\mu\nu}\widetilde{B}_{\mu\nu} + \cdots.$$

Here s_0 is independent of scale, while the $\partial_\mu K_\mu$ anomaly coefficient is defined as the scale-dependent function $k(g(\mu))$. In the $\langle\partial_\mu R_\mu\rangle$ equation "+..." indicates the non-local contribution of the internal anomaly term of (2.10). There is a similar non-local contribution to $\langle\partial_\mu K_\mu\rangle$, which cancels that of $\langle\partial_\mu R_\mu\rangle$ in the linear combination (2.4) that gives $\langle\partial_\mu S_\mu\rangle$. These contributions are irrelevant for our analysis, since the possible local terms they contain are $O(\beta(g(\mu)))$.

From (2.4) we see that the local terms in (2.23) satisfy

$$\widetilde{b}(g) + \left(1 - \frac{3N_c}{N_f} - \gamma(g)\right)\tilde{k}(g) = -s_0. \tag{2.24}$$

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In applications to asymptotically-free electric supersymmetric QCD, the coupling $g(\mu)$ vanishes in the ultraviolet, and so does $\gamma(g)$; thus the ultraviolet contributions $\tilde{b}_{UV}$ and $\tilde{k}_{UV}$ can be easily obtained from 1-loop contributions to the relevant 3-point correlators. These are normalized so that each quark or anti-quark field contributes $1/N_c$ to the quantity $\tilde{k}_{UV}$, when J is the baryon current of (2.1). If we equate the values of the left side of (2.24) at scale μ and in the UV limit, we find

$$\widetilde{b}(g) = b_{UV} + \gamma(g)\tilde{k}_{UV} - \left(1 - \frac{3N_c}{N_f} - \gamma(g)\right)\left[\tilde{k}(g) - \tilde{k}_{UV}\right]. \tag{2.25}$$

This result can be checked against the explicit 2-loop calculation [9] of the flavor central function $b(g)$ and it agrees. We do not discuss this here, because we are more interested in non-perturbative application to electric SUSY QCD in the conformal window, $3N_c/2 < N_f < 3N_c$, where there is evidence that the theory flows to a non-trivial fixed point g_* [2]. Although g_* can only be calculated at the weakly coupled end of the conformal window (where $N_f = 3N_c(1-\epsilon)$) we know that the β-function (2.5) vanishes, and this is enough to give the exact infrared limit of the anomalous dimension

$$\gamma_{IR} = \left(1 - \frac{3N_c}{N_f}\right). \tag{2.26}$$

Then (2.25) becomes

$$b_{IR} - b_{UV} = \gamma_{IR}\tilde{k}_{UV}, \tag{2.27}$$

which is our first non-perturbative result for the flow of a flavor central charge. Since gauginos do not contribute to flavor current correlators and the quark and anti-quark contributions to the combination $R + K/3$ cancel, it follows that $b_{UV} = -\tilde{k}_{UV}$ for correlators of all of the flavor currents in (2.1). For the baryon current, $b_{UV} = 2N_f/N_c$ and we have the total flow

$$b_{IR} - b_{UV} = 6\left(1 - \frac{N_f}{3N_c}\right), \tag{2.28}$$

which is positive in the entire conformal window, contrary to c-theorem intuition.

We now turn our attention to magnetic supersymmetric QCD with gauge group $SU(N_c')$, with $N_c' = N_f - N_c$. The matter content consists of N_f flavors of magnetic quark and anti-quark superfields $q_i^\alpha(x,\theta)$, $\tilde{q}_\alpha^i(x,\theta)$ plus the gauge neutral $N_f \times N_f$ meson superfield M_j^i. There are conventional gauge interactions and a cubic superpotential $W = fqM\tilde{q}$. The dual theory has the same flavor group $SU(N_f)_q \times SU(N_f)_{\tilde{q}} \times U(1)_B$, but for simplicity we shall consider only the baryon current [4]

$$J_\mu = \frac{1}{N_c'}\frac{1}{2}\left(\bar{\psi}_q\gamma_\mu\gamma_5\psi_q - \bar{\psi}_{\tilde{q}}\gamma_\mu\gamma_5\psi_{\tilde{q}}\right). \tag{2.29}$$

We also need the separate Konishi currents of the quarks and the mesons

$$K_\mu^{(q)} = \frac{1}{2}\left(\bar{\psi}_q\gamma_\mu\gamma_5\psi_q + \bar{\psi}_{\tilde{q}}\gamma_\mu\gamma_5\psi_{\tilde{q}}\right), \tag{2.30}$$

$$K_\mu^{(M)} = \frac{1}{2}\text{Tr}\bar{\psi}_M\gamma_\mu\gamma_5\psi_M,$$

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[4]For simplicity we do not write the scalar contributions to the currents in eqs.(2.29–2.31).

while the fermion content of the R current is

$$R_\mu = \frac{1}{2}\bar{\lambda}^a\gamma_\mu\gamma_5\lambda^a - \frac{1}{3}(K_\mu^{(q)} + K_\mu^{(M)}).$$ (2.31)

The divergences $\partial_\mu K_\mu^{(q)}$ and $\partial_\mu K_\mu^{(M)}$ have classical contributions, since the superpotential is not invariant under the relevant $U(1)$ transformations, and $\partial_\mu K_\mu^{(q)}$ also has an anomaly. The internal R-current anomaly also involves $F\tilde{F}$ and there is a superpotential contribution [10, 18]. It is significant for our analysis that there is a unique combination [10] of these currents which is classically conserved and anomaly-free, namely

$$S_\mu = R_\mu + \frac{1}{3}\left(1 - \frac{3N_c'}{N_f} - \gamma_q\right)\left(K_\mu^{(q)} - 2K_\mu^{(M)}\right) - \frac{1}{3}\left(2\gamma_q + \gamma_M\right)K_\mu^{(M)}$$ (2.32)

The combination $K_\mu^{(q)} - 2K_\mu^{(M)}$ is conserved classically. Its coefficient in (2.32) is again the numerator of the NSVZ gauge β-function (2.5), and the coefficient of $K_\mu^{(M)}$ is essentially the Yukawa β-function $\beta_f = f(2\gamma_q + \gamma_M)$.

The previous analysis must be generalized to include a superpotential. There are only trivial changes in the derivation of the relation between the external trace anomaly $\tilde{b}$ and the central function $b(g(1/x), f(1/x))$, and the result that they coincide at fixed points is unchanged. The analysis of the flow of flavor central functions is also easily repeated. The third equation of (2.23) is replaced by the pair

$$\langle\partial_\mu K_\mu^{(q)}\rangle = -\frac{1}{16\pi^2}\tilde{k}^{(q)}\left(g(\mu), f(\mu)\right)B_{\mu\nu}\tilde{B}_{\mu\nu} + ...$$ (2.33)

$$\langle\partial_\mu K_\mu^{(M)}\rangle = -\frac{1}{16\pi^2}\tilde{k}^{(M)}\left(g(\mu), f(\mu)\right)B_{\mu\nu}\tilde{B}_{\mu\nu} + ...$$

and one finds

$$\tilde{b} = b_{UV} + \gamma_q\tilde{k}_{UV}^{(q)} + \gamma_M\tilde{k}_{UV}^{(M)}$$ (2.34)

$$-\left(1 - \frac{3N_c'}{N_f} - \gamma_q\right)\left(\tilde{k}^{(q)} - 2\tilde{k}^{(M)} - \tilde{k}_{UV}^{(q)} + 2\tilde{k}_{UV}^{(M)}\right)$$

$$+ (2\gamma_q + \gamma_M)\left(\tilde{k}^{(M)} - \tilde{k}_{UV}^{(M)}\right)$$

in which the subscript UV denotes a quantity evaluated in the ultraviolet from 1-loop graphs, and other quantities are evaluated at scale μ. One can also check, as in the electric case, that this formula agrees with the two-loop calculation.

We now go to the IR fixed point where the coefficients of the last two terms of (2.34) vanish, since they are related to β_g and β_f, respectively. This gives the IR values of the anomalous dimensions

$$\gamma_q^{IR} = -\frac{1}{2}\gamma_M^{IR} = \left(1 - \frac{3N_c'}{N_f}\right)$$ (2.35)

and we obtain from (2.34)

$$b_{IR} - b_{UV} = \left(\gamma_q^{IR} \tilde{k}_{UV}^{(q)} + \gamma_M^{IR} \tilde{k}_{UV}^{(M)} \right), \tag{2.36}$$

which is a non-perturbative formula for the flow of flavor charges in the magnetic theory. For the baryon current, $b_{UV} = -\tilde{k}_{UV}^{(q)} = 2N_c/N_c'$ and $\tilde{k}_{UV}^{(M)} = 0$, so the flow of the baryon central charge is

$$b_{IR} - b_{UV} = 6 \left(1 - \frac{N_f}{3N_c'} \right) = 2 \frac{2N_f - 3N_c}{N_f - N_c} \tag{2.37}$$

which is again positive throughout the conformal window. From (2.28) and (2.37) one can check the equality of the IR values of the central function, namely $b_{IR} = 6$. This is not a new confirmation of duality, since it is the same 't Hooft anomaly matched in electric-magnetic duality [2].

We briefly discuss the question of constancy of the flavor central charge on conformal fixed lines [18], confining our attention to the $N = 4$ and $N = 2$ cases. The matter content of the $N = 4$ theory is three adjoint chiral multiplets, while for $N = 2$, with $G = SU(N_c)$, $N_f = 2N_c$ fundamental hypermultiplets are required to produce a fixed line. In each case there is a particular cubic superpotential, and the condition for the fixed line is a linear relation between g and the Yukawa coupling f. It is straightforward to generalize the previous derivations and show that formulas of the type (2.27) and (2.36) hold for the $N = 4$ and $N = 2$ cases, respectively, with UV-subscripted quantities calculated from the relevant 1-loop graphs. To establish marginal constancy at the non-perturbative level, we need only observe [18] that anomalous dimensions vanish on the fixed line, so we have $b =$ constant. At first thought one might expect that the value of the central charge depends on the single coupling, say g, that remains on the fixed line; however we see that b is determined by the free field content of the theory.

An even simpler argument is to note [10] that S_μ and R_μ coincide at infrared fixed points or fixed lines. But S_μ is anomaly-free, so we have $\langle \partial_\mu S_\mu J_\nu J_\rho \rangle_{UV} = \langle \partial_\mu S_\mu J_\nu J_\rho \rangle_{IR} = \langle \partial_\mu R_\mu J_\nu J_\rho \rangle_{IR}$. Hence, the central charge b on an infrared fixed line is equal to the anomaly coefficient $-s_0$ and thus depends only on the free-field content of the theory and not on any coupling constant.

3   An alternative approach

In this section we compute the flavor central charges using a somewhat different approach. This method uses the correlation function $b(g(1/x))$ of (2.6) in a more direct way, and does not require an explicit regularization of the singularity at $x = 0$. This simplifies the analysis. The method is quite general, but we work with the baryon number current of the electric SUSY QCD, given in (2.1) for simplicity.

Let us consider the correlator (2.6) again and compute the scale derivative $\mu \partial/\partial\mu \langle J_\mu(x) J_\nu(0) \rangle$ at fixed bare coupling $g_0 = g(\mu)$, which gives by definition the correlator $\langle J_\mu(x) J_\nu(0) \int d^4z\, \Theta(z) \rangle$. In the presence of an external field B_μ coupled to J_μ and in quadratic approximation the matrix

element $< \int d^4z \ \Theta(z) >$ has the form

$$< \int d^4z \ \Theta(z) > = \int \frac{d^4k}{(2\pi)^4} \ f(g(k)) \cdot \frac{1}{4} B^{\mu\nu}(k) B_{\mu\nu}(-k) = \tag{3.1}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{4} B^{\mu\nu}(k) B_{\mu\nu}(-k) \left(f_0 - \frac{1}{16\pi^4} \int d^4x \mathrm{e}^{ikx} \frac{1}{x^4} \mu \frac{\partial}{\partial \mu} b(g(1/x)) \right) =$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{4} B^{\mu\nu}(k) B_{\mu\nu}(-k) \left(f_0 - \frac{1}{8\pi^4} \int d^4x \mathrm{e}^{ikx} \frac{1}{x^4} \frac{\partial}{\partial \log x^2} b(g(1/x)) \right) .$$

The above integral is convergent at $x = 0$ because of asymptotic freedom and the RG equations. In eq. (3.1) we explicitly introduced a constant f_0 associated with the regularization of the correlator (2.6) at $x = 0$. This contact term is irrelevant for the present analysis. However it can be easily determined. Indeed, in the UV limit, i.e. $|k| \to \infty$, the x-integral in eq. (3.1) vanishes, and hence $f_0 = f_{UV} = f(g(k)) \ |_{|k| \to \infty}$. In particular in the case of the baryonic current $f_0 = N_f/4\pi^2 N_c$.

The IR limit corresponds to $|k| \to 0$, i.e. $f_{IR} = f(g(k)) \ |_{|k| \to 0}$. Thus from (3.1) one gets

$$f_{UV} - f_{IR} = -\frac{1}{8\pi^2} \int_0^\infty d\log x^2 \ \frac{\partial}{\partial \log x^2} b(g(1/x)) = \frac{1}{8\pi^2} (b_{UV} - b_{IR}). \tag{3.2}$$

We turn now to the computation of the function $f(g(k))$. The simplest way is to observe that the $B_\mu J_\mu$ coupling is essentially that of SQED, with external "photon" field B_μ and coupling constant e. In SQED the conformal anomaly can be computed as a scale derivative $\mu \partial/\partial \mu$ (at fixed values of $g(\mu), \ e(\mu)$) of the effective action for the external field B_μ

$$\Gamma[B] = \int \frac{d^4k}{(2\pi)^4} \ \frac{1}{4e^2(k)} B_{\mu\nu}(k) B^{\mu\nu}(-k), \tag{3.3}$$

where $e(k)$ is the effective running coupling constant. We have

$$< \int d^4z \ \Theta(z) > = \mu \frac{\partial}{\partial \mu} \Gamma[B] = \frac{1}{N_c^2} \int \frac{d^4k}{(2\pi)^4} B(k)^{\mu\nu} B_{\mu\nu}(-k) \cdot \frac{\beta_e(e(k), g(k))}{2e^3(k)}$$

$$= \frac{1}{N_c^2} \int \frac{d^4k}{(2\pi)^4} B(k)^{\mu\nu} B_{\mu\nu}(-k) \cdot \frac{N_f N_c}{16\pi^2} \ [1 - \gamma(g(k))]. \tag{3.4}$$

This expression follows from the general result for the N=1 supersymmetric QED β-function $\beta_e = n (1 - \gamma) e^3/8\pi^2$ given in ref. [11], γ being the anomalous dimension of the matter chiral superfields and n standing for the number of flavors. In the present context we have $n = N_c N_f$, and $e \to 0$ because the QED gauge field B_μ is external. Therefore,

$$f(g(k)) = \frac{N_f}{4\pi^2 N_c} \ [1 - \gamma(g(k))]. \tag{3.5}$$

gives an exact formula for $f(g(k))$ as a linear function of the anomalous dimension. (The relation (3.5) is not scheme independent, but rather holds in the scheme of [10].) The inverse Fourier transform of (3.5) then gives $\mu \frac{\partial}{\partial \mu} \left(b(g(1/x))/x^4 \right)$, and $b(g(1/x))$ can be easily computed.

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Since $\gamma(g(k)) \mid_{k \to \infty} = 0$ and $\gamma(g(k)) \mid_{k \to 0} = 1 - 3N_c/N_f$, from eq. (3.5) immediately follows that

$$b_{IR} - b_{UV} = 8\pi^2(f_{IR} - f_{UV}) = 2 \cdot \frac{N_f}{N_c} \cdot \left(\frac{3N_c}{N_f} - 1 \right). \tag{3.6}$$

We therefore recover the result of Section 2. Note that we did not use the current S_μ of (2.4).

It is worth observing that the SQED β-function is also implicitly present in the treatment of Section 2. To show this we consider a theory with gauge group extended to $SU(N_c) \times U(1)$ with $U(1)$ coupling constant e. This theory has the operator trace anomaly

$$\Theta = -\frac{3N_c - N_f(1 - \gamma)}{32\pi^2} \left(F_{\mu\nu}^a \right)^2 + \frac{N_c N_f(1 - \gamma)}{16\pi^2 N_c^2} \left(B_{\mu\nu} \right)^2, \tag{3.7}$$

where the coefficient of $(B_{\mu\nu})^2$ is $\beta_e/2e^3 N_c^2$ and $\gamma = \gamma(e(\mu), g(\mu))$. In the limit $e \to 0$ we can compare (3.5) and (3.7) with (2.9) and identify

$$q = \lim_{e \to 0} \frac{2\beta_e}{e^3 N_c^2} = \frac{N_c N_f(1 - \gamma(0, g(\mu)))}{4\pi^2 N_c^2} = f(g(\mu)) \tag{3.8}$$

It is instructive to compare (3.1) and (2.21). Consider first (2.21). The correlator $\langle J_\mu J_\nu \int \Theta \rangle$ is renormalization group invariant. In momentum space the r.h.s of (2.21) may depend only on the running coupling $g(k)$, but not on $g(\mu)$. Therefore, the μ-dependence of q is compensated by that of the correlator $\langle J_\mu J_\nu \int (F_{\mu\nu}^a)^2 \rangle$ which has a perturbative expansion starting with $g^4(\mu) \log k^2/\mu^2$. On the other hand the form of (3.1) is explicitly renormalization group invariant. The local part q of (2.21) was identified above as $q(g(\mu)) = f(g(\mu))$. The non-local part of (3.1) corresponding to the contribution of $\langle J_\mu J_\nu \int (F_{\mu\nu}^a)^2 \rangle$ in (2.21), is then

$$f(g(k)) - f(g(\mu)) \sim [3N_c - N_f(1 - \gamma(g(\mu)))] \cdot [g^4(\mu) \log k^2/\mu^2 + \ldots]. \tag{3.9}$$

This is proportional to $\beta(g(\mu))$ because of the RG equation

$$\mu \frac{\partial}{\partial \mu} b(g(1/x)) = \beta(g(\mu)) \frac{\partial}{\partial g(\mu)} b(g(1/x)) \tag{3.10}$$

which can be used in (3.1), and perturbation theory gives the leading power $g^4(\mu)$. In this way we establish a correspondence between the approaches of Section 2 and the present section.

Another method of calculation is to consider first the anomalous superspace operator equations in the absence of the external B_μ field (see, for example [11, 5])

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} - \frac{N_f}{48\pi^2} \left(1 - \frac{3N_c}{N_f} - \gamma(g) \right) D_\alpha W^2 = 0, \quad \bar{D}^2 K - \frac{N_f}{2\pi^2} W^2 = 0, \tag{3.11}$$

where $J_{\alpha\dot\alpha}$ is the supercurrent, K is the Konishi operator. By combining eqs. (3.11) one can get

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} - \frac{1}{24} \left(1 - \frac{3N_c}{N_f} - \gamma(g) \right) D_\alpha \bar{D}^2 K = 0,$$

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which is the superspace version of (2.4). In the presence of the external field B_μ the operator $\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} - \frac{1}{24}\left(1 - \frac{3N_c}{N_f} - \gamma(g)\right) D_\alpha \bar{D}^2 K$ does not vanish but is proportional to the unit operator times a local functional of the external field. Hence, for the component matrix element containing the stress tensor we have ($\Theta \propto [D^\alpha, \bar{D}^{\dot\alpha}]J_{\alpha\dot\alpha}|_{\theta=0}$, $L_{matter} \propto \{D^2, \bar{D}^2\}K\}_{\theta=0}$)

$$< \int d^4z \left[\Theta(z) - \left(1 - \frac{3N_c}{N_f} - \gamma(g)\right) L_{matter}(z)\right] >= u_0 \int \frac{d^4k}{(2\pi)^4} \frac{1}{4} B(k)^{\mu\nu} B_{\mu\nu}(-k). \quad (3.12)$$

The constant u_0 does not depend on the external field. It is also independent of the renormalization group scale μ since the operator on the left hand side is renormalization group invariant.

(It is worth noting that the above statement is a consequence of a general result. We refer to an operator O that has vanishing matrix elements between any Fock physical states as a *null* operator. This is described by the operator equation $O = 0$. Thus, $\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} - \frac{1}{24}(1 - 3N_c/N_f - \gamma)D_\alpha \bar{D}^2 K$ is a null operator. In the presence of an external field a null operator may not vanish. In general the r.h.s. of the operator equation may be a linear combination of various local operators O_i with the coefficients y_i being non-trivial local functionals of the external field, i.e. $O = \sum_i y_i O_i$. In the simplest case the O operator mixes only with the unit operator, i.e. the operator equation reads $O = y \cdot \mathbf{1}$. In such a case the matrix element of O is just a local functional y of the external field. Note that the one-loop form of the 't Hooft external anomaly for a current J_μ is a particular consequence of this fact. Indeed, assuming that J_μ does not have any internal anomaly its anomalous dimension is vanishing. Therefore the matrix element of $\partial_\mu J_\mu$ is renormalization group invariant, i.e. it may depend only on the running coupling $g(k)$, where k is a momentum of the external field. On the other hand, as mentioned above, y is a local functional of the external field, and hence it does not depend on $g(k)$. Therefore the matrix element in question is exactly one-loop.)

For technical reasons it is convenient to consider a particular momentum mode k and define the reduced matrix elements $\ll \cdots \gg$ obtained by dividing the conventional matrix elements $\langle \cdots \rangle$ by $(1/4)B^{\mu\nu}(k)B_{\mu\nu}(-k)$. The reduced matrix element of the operator $\int d^4z \left[\Theta(z) - (1 - 3N_c/N_f - \gamma(g)\right) L_{matter}(z)]$ (which is just equal to u_0) does not change under the renormalization group flow with respect to k^2. Therefore we have

$$\Delta f = f(g(k))\,|_{|k|\to 0} - \ f(g(k))\,|_{|k|\to\infty} = \ll \int d^4z \ (1 - 3N_c/N_f - \gamma(g(\mu)))L_{matter}(z) \gg|_{|k|\to 0} -$$
$$\ll \int d^4z (1 - 3N_c/N_f - \gamma(g(\mu)))L_{matter}(z) \gg|_{|k|\to\infty}. \quad (3.13)$$

The matrix element $\ll \int d^4z \ (1 - 3N_c/N_f - \gamma(g)) L_{matter}(z) \gg$ is proportional to $1 - 3N_c/N_f - \gamma(g(k))$ [5] and hence vanishes in the infrared, i.e. at $k^2 \to 0$. Thus we have, precisely as before,

$$\Delta f = - \ll \int d^4z \ (1 - 3N_c/N_f - \gamma(g)) L_{matter}(z) \gg|_{|k|\to\infty} =$$
$$= \frac{1}{4\pi^2}\left(\frac{3N_c}{N_f} - 1\right)\frac{N_f}{N_c}, \quad (3.14)$$

where the result follows from the identification of L_{matter} with the $\theta^2\bar\theta^2$ component of the Konishi superfield, and the (one-loop) Konishi anomaly.

The story becomes more complicated for the gravitational central charges which are considered in section 4. In this case one may try to analyse the matrix element $< \int d^4z\ \Theta(z) >$ of the stress tensor in the presence of an external gauge field V_μ field coupled to the R_μ current in order to compute the flow, say, of the c charge. However, the coupling of the R_μ current to the "QED" vector field V_μ breaks supersymmetry and this "QED" β-function is not simply related to the anomalous dimensions γ. A direct way to compute the flows of the c and a charges is to consider the matrix element of the divergence of the R_μ current in the presence of an external gauge field coupled to the R_μ current. In this way it is easy to rephrase the considerations of Section 4 in the formalism of this section.

4   Gravitational central functions

As discussed in [8] the correlator of two stress tensors can be written in the form

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{1}{48\pi^4}\Pi_{\mu\nu\rho\sigma}\frac{c(g(1/x))}{x^4} + \pi_{\mu\nu}\pi_{\rho\sigma}\frac{f(\ln x\mu, g(1/x))}{x^4}, \qquad (4.1)$$

where $\pi_{\mu\nu} = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)$ and $\Pi_{\mu\nu\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - 3(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$ is the transverse traceless spin 2 projector. One of the objects of primary concern in this section is the flow of the central function $c(g(1/x))$ that we will relate to the coefficient of the square of the Weyl tensor in the external gravitational trace anomaly. The second object of interest is the coefficient a of the Euler term in the trace anomaly. We will give non-perturbative formulas for the flows of these quantities. The internal trace anomaly is responsible for the second tensor structure in (4.1). It is proportional to $\beta(g(\mu))$, and thus vanishes at critical points.

Since $T_{\mu\nu}(x)$ and $R_\mu(x)$ are both components of the supercurrent superfield $J_{\alpha\dot\alpha}(z)$, $z \equiv (x, \theta, \bar\theta)$, there is a relation between the $\langle TT \rangle$ and $\langle RR \rangle$ correlators. For a critical supersymmetric theory, they are both contained in the supercorrelator

$$\langle J_{\alpha\dot\alpha}(z)J_{\beta\dot\beta}(0)\rangle \sim c\,\frac{s_{\alpha\dot\beta}\bar s_{\beta\dot\alpha}}{(s^2\bar s^2)^2} \qquad (4.2)$$

given in Section 5 of [5] where the notation is explained. This means that the central charges of the TT and RR OPE's are given by the same constant c which is a fixed point value of $c(g(\mu))$. Off criticality there is a more complicated relation between $\langle TT \rangle$ and $\langle RR \rangle$ that is not required for our work.

In Section 2 we introduced a source for the flavor current $J_\mu(x)$ in order to show that the coefficient of the external trace anomaly coincides at fixed points with the central function $b(g(1/x))$ and related the trace and $\partial_\mu R_\mu$ anomalies using global supersymmetry. In the present case sources for $T_{\mu\nu}$ and R_μ take us out of the realm of global supersymmetry and require external field supergravity where the anomaly situation is more complex, as we now discuss.

We introduce the background metric $g_{\mu\nu}(x)$ and source $V_\mu(x)$ for the R-current. In these background fields the trace anomaly of a critical supersymmetric theory has the form

$$\Theta = \frac{c}{16\pi^2}\,(W_{\mu\nu\rho\sigma})^2 - \frac{a}{16\pi^2}\,\widetilde{R}_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + \frac{c}{6\pi^2}\,V_{\mu\nu}^2\;. \tag{4.3}$$

Here $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and $\widetilde{R}_{\mu\nu\rho\sigma}$ is the dual of the curvature tensor, the second term being the Euler density; $V_{\mu\nu}$ is the field strength of V_μ. The coefficients of the $(W_{\mu\nu\rho\sigma})^2$ and $(V_{\mu\nu})^2$ terms are related while that of the Euler density is an independent constant. In a free supersymmetric gauge theory with N_v gauge and N_χ chiral multiplets, the constants are [19]

$$c_{UV} = \frac{1}{24}\,(3N_v + N_\chi)\quad,\quad a_{UV} = \frac{1}{48}\,(9N_v + N_\chi)\;. \tag{4.4}$$

Off criticality there are additional terms in Θ that are proportional to $\beta(g(\mu))$ and do not contribute to the total flow, and the central charges depend on the coupling, i.e. $c = c(g(\mu))$ and $a = a(g(\mu))$.

In the superspace description of these anomalies, the external metric (actually the vierbein $e_\mu{}^a$) and current source (the supergravity axial vector auxiliary field) are contained in a single superfield $H^a(x,\theta,\bar{\theta})$, and the trace anomaly and the $\partial_\mu R_\mu$ anomaly are components of the superfield equation

$$\bar{D}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} = D_\alpha J \tag{4.5}$$

where the chiral superfield J (the supertrace) has the form

$$J = \frac{1}{24\pi^2}(cW^2 - a\Xi_c) \tag{4.6}$$

in terms of the *super*Weyl tensor $W_{\alpha\beta\gamma}$ and the chirally projected *super*Euler density (see Appendix A for details).

We need the $\partial_\mu R_\mu$ anomaly. Although it is fairly straightforward to compute it as the theta component of (4.5), and we will do this in Appendix A, we present here an indirect argument that uses the general structure of (4.6) but does not require any superspace technology. The $\partial_\mu R_\mu$ anomaly will have the form $\partial_\mu R_\mu = (uc + va)R_{abcd}\widetilde{R}^{abcd} + (wc + za)V\widetilde{V}$ with model independent coefficients u, v, w and z. It is thus sufficient to compute in a free supersymmetric gauge theory in order to find these coefficients. Because of the ratio $-1/3$ between the R-charges of gauginos and matter fermions, we know that $uc+va$ can be obtained from the $\langle TTR\rangle$ triangle graph as a pure numerical (i.e. $1/24\pi^2$) multiple of $3N_v - N_\chi$, and $wc + za$ can be obtained from the $\langle RRR\rangle$ triangle as a numerical multiple of $27N_v - N_\chi$. Using the values for c_{UV} and a_{UV} in (4.4) we find

$$\partial_\mu(\sqrt{g}R^\mu) = \frac{c-a}{24\pi^2}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + \frac{5a - 3c}{9\pi^2}V_{\mu\nu}\widetilde{V}^{\mu\nu} \tag{4.7}$$

The next step is to note that the W^2 anomaly coefficient of a non-critical interacting theory, $\tilde{c}(g(\mu))$, can be related to the central function $c(g(1/x))$ by the same argument as in Section 2, which in fact is exactly the argument given in [8]. (The Euler term of Θ gives no contribution

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to the integrated trace anomaly.) The functions $\tilde{c}(g(\mu))$ and $c(g(\mu))$ coincide at fixed points of the flow.

We now proceed to the calculations of c and a in the electric $SU(N_c)$ SQCD, where $N_v = N_c^2 - 1$ and $N_\chi = 2N_cN_f$. To achieve this goal, we can apply arguments of Section 2 to calculate the flow of the combinations of anomaly coefficients $\tilde{c}(g(\mu)) - a(g(\mu))$ and $5a(g(\mu)) - 3\tilde{c}(g(\mu))$ from the three-point correlators $\langle TTR \rangle$ and $\langle RRR \rangle$, respectively.

The anomalies of the correlators $\langle TTS \rangle$, $\langle TTR \rangle$, and $\langle TTK \rangle$ are summarized by the equations

$$\langle \partial_\mu(\sqrt{g}S^\mu) \rangle = \frac{1}{12\pi^2} s_1 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}^{\lambda\tau},$$

$$\langle \partial_\mu(\sqrt{g}R^\mu) \rangle = \frac{1}{12\pi^2} \left(\tilde{c}(g(\mu)) - a(g(\mu)) \right) \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}^{\lambda\tau},$$

$$\langle \partial_\mu(\sqrt{g}K^\mu) \rangle = \frac{1}{12\pi^2} k(g(\mu)) \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}^{\lambda\tau}, \tag{4.8}$$

where we omitted the non-local $O(\beta(g(\mu)))$ terms. In the ultraviolet limit the quantity $k(g(\mu))$ can be obtained from the 1-loop triangle graph as $k \to k_{UV} = -N_f N_c/8$. Similarly $\tilde{c}(g(\mu)) - a(g(\mu)) \to c_{UV} - a_{UV} = -\frac{1}{16}(N_c^2 - 1 - \frac{2}{3}N_f N_c)$. It is now immediate to write the analogue of (2.25) which is

$$\tilde{c}(g) - a(g) = c_{UV} - a_{UV} + \frac{1}{3}\gamma(g)k_{UV} - \frac{1}{3}\left(1 - \frac{3N_c}{N_f} - \gamma(g)\right)(k(g) - k_{UV}). \tag{4.9}$$

To discuss the $\langle RRR \rangle$ anomaly, we observe that the amplitude of the triangle graph for the contribution of one Majorana spinor with current $J_\mu = \frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi$ has the Bose-symmetric anomaly

$$\frac{\partial}{\partial z_\rho}\langle J_\mu(x)J_\nu(y)J_\rho(z)\rangle_{UV} = -\frac{1}{12\pi^2}\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial y_\sigma}\delta(x-z)\delta(y-z) \equiv \frac{16}{9}\mathcal{A}_{\mu\nu}(x,y,z) \tag{4.10}$$

(see [21] for a recent discussion of the anomaly in x-space). We then write $R_\mu = S_\mu + \frac{1}{3}(\gamma - \gamma_{IR})K_\mu$ where S_μ is the internal anomaly-free current (2.4). The various Bose symmetric contributions to $\langle RRR \rangle$ then have anomalous divergences that can be written as:

$$\frac{\partial}{\partial z_\rho}\langle R_\mu R_\nu R_\rho \rangle = [5a(g(\mu)) - 3c(g(\mu)]\mathcal{A}_{\mu\nu}$$

$$\frac{\partial}{\partial z_\rho}\langle S_\mu S_\nu S_\rho \rangle = s_3\,\mathcal{A}_{\mu\nu}$$

$$\frac{\partial}{\partial z_\rho}[\langle S_\mu S_\nu K_\rho \rangle + \langle S_\mu K_\nu S_\rho \rangle + \langle K_\mu S_\nu S_\rho \rangle] = 3k_1(g(\mu))\,\mathcal{A}_{\mu\nu}$$

$$\frac{\partial}{\partial z_\rho}[\langle S_\mu K_\nu S_\rho \rangle + \langle K_\mu K_\nu S_\rho \rangle + \langle K_\mu S_\nu K_\rho \rangle] = 3k_2(g(\mu))\mathcal{A}_{\mu\nu}$$

$$\frac{\partial}{\partial z_\rho}\langle K_\mu K_\nu K_\rho \rangle = k_3(g(\mu))\mathcal{A}_{\mu\nu} \tag{4.11}$$

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where the tensor indices μ, ν, ρ, are associated with coordinates x, y, z, respectively; we again omitted $O(\beta)$ irrelevant nonlocal terms. The anomaly coefficient s_3 is scale independent while other anomaly coefficients depend on $g(\mu)$.

Proceeding as before and evaluating s_3 in the UV limit from triangle graphs, we obtain

$$5a - 3c = 5a_{UV} - 3c_{UV} + h, \tag{4.12}$$

where

$$h = \gamma_{IR}k_{1\,UV} - \gamma_{IR}^2\frac{1}{3}k_{2\,UV} + \gamma_{IR}^3\frac{1}{27}k_{3\,UV} + (\gamma - \gamma_{IR})k_1$$
$$+ (\gamma - \gamma_{IR})^2\frac{1}{3}k_2 + (\gamma - \gamma_{IR})^3\frac{1}{27}k_3. \tag{4.13}$$

The subscript UV indicates the free field value, while the other quantities are evaluated at the scale μ. We have

$$k_{UV} = -\frac{1}{16}N_\chi, \qquad k_{3\,UV} = \frac{9}{16}N_\chi, \qquad k_{2UV} \equiv -\frac{9}{16}N_\chi\frac{N_c}{N_f}, \qquad k_{1\,UV} = \frac{9}{16}N_\chi\left(\frac{N_c}{N_f}\right)^2. \tag{4.14}$$

Finally, collecting (4.9) and (4.12), we find the non-perturbative results

$$c = c_{UV} + \frac{5}{6}(\gamma - \gamma_{IR})k + \frac{5}{6}\gamma_{IR}k_{UV} + \frac{1}{2}h,$$
$$a = a_{UV} + \frac{1}{2}(\gamma - \gamma_{IR})k + \frac{1}{2}\gamma_{IR}k_{UV} + \frac{1}{2}h. \tag{4.15}$$

Let us compare our results with two-loop calculations. The work of [6] shows that $a(g)$ has no 2-loop corrections and in [7] the result

$$c(g) = \frac{1}{24}\left(3N_v + N_\gamma - \gamma_i^i + N_v\frac{\beta(g)}{g}\right) \tag{4.16}$$

for c was obtained. We see that the two-loop correction is the sum of a contribution proportional to γ and a contribution proportional to $\beta(g)/g$. Formulas (4.15) show that the coefficients of the γ terms are

$$\frac{5}{6}k_{UV} + \frac{1}{2}k_{1\,UV} - \frac{1}{3}\gamma_{IR}k_{2\,UV} + \frac{1}{18}\gamma_{IR}^2k_{3\,UV} = -\frac{1}{24}N_cN_f,$$
$$\frac{1}{2}k_{UV} + \frac{1}{2}k_{1\,UV} - \frac{1}{3}\gamma_{IR}k_{2\,UV} + \frac{1}{18}\gamma_{IR}^2k_{3\,UV} = 0, \tag{4.17}$$

for c and a, respectively, in agreement with the two-loop results of [6] and (4.16). Note that $\gamma_i^i = N_cN_f\gamma$ in the present notation and γ is the anomalous dimension of the gauge invariant $\tilde{Q}Q$ which is twice the value of the field anomalous dimension used in [7]. The comparison of the coefficient of the $\beta(g)/g$ terms, instead, is more subtle and requires the precise knowledge of the two-loop corrections to the functions k, k_1, k_2 and k_3. For example, a two-loop correction to k is expected from the photon chiral anomaly in an external gravitational field [20].

In the infrared limit $\gamma - \gamma_{IR}$ vanishes (it is the numerator of $\beta(g)$) and the central charge flows are

$$c_{IR} - c_{UV} = \frac{N_c N_f}{48}\gamma_{IR}\left(3\frac{N_c}{N_f} + 9\frac{N_c^2}{N_f^2} - 4\right), \quad a_{IR} - a_{UV} = -\frac{N_c N_f}{48}\gamma_{IR}^2\left(2 + 3\frac{N_c}{N_f}\right). \quad (4.18)$$

We can check agreement with the two loop results also from these formulas, but only in the weakly coupled region $\gamma_{IR} \ll 1$. Again, the coefficients of γ_{IR} are the expected ones. The fact that a is two-loop uncorrected is exhibited by the appearance of γ_{IR}^2 in the expression of $a_{IR} - a_{UV}$. The term proportional to $\beta(g)/g$ in (4.16), on the other hand, cannot be reproduced in the flows (4.18): although it is nonvanishing at the order g^2, it is cancelled at criticality by the higher loop corrections.

$$*\quad *\quad *$$

In the considerations above we have discussed the central functions along the RG flows. If one is only interested in the fixed point values of c and a , one can use a shortcut because the coefficient $(1 - 3N_c/N_f - \gamma)$ vanishes at the IR fixed point, so R_μ and S_μ effectively coincide there. It is again crucial that the internal anomaly-free current S_μ of (2.4) has one-loop exact external anomalies, so that the IR values of the R_μ-anomalies coincide with the UV values of the corresponding S_μ-anomalies, which makes them computable. Thus, for example,

$$\frac{\partial}{\partial x_\mu}\langle R_\mu(x)R_\nu(y)R_\rho(z)\rangle_{IR} = \frac{\partial}{\partial x_\mu}\langle S_\mu(x)S_\nu(y)S_\rho(z)\rangle_{IR}$$
$$= \frac{\partial}{\partial x_\mu}\langle S_\mu(x)S_\nu(y)S_\rho(z)\rangle_{UV} \quad (4.19)$$

and the last anomaly can be obtained from the one-loop graphs of $S_\mu^{UV} = R_\mu + \frac{1}{3}(1 - 3N_c/N_f)K_\mu$ with R_μ and K_μ defined in (2.2). We find

$$5a_{IR} - 3c_{IR} = \frac{9}{16}\left(N_c^2 - 1 - 2N_c N_f\left(\frac{N_c}{N_f}\right)^3\right) \quad (4.20)$$

In the UV limit, $\frac{\partial}{\partial x_\mu}\langle R(x)_\mu R(y)_\nu R(z)_\rho\rangle$ is determined by the one-loop fermion triangle graphs of R_μ in (2.2) which yield

$$5a_{UV} - 3c_{UV} = \frac{9}{16}\left(N_c^2 - 1 - \frac{2N_c N_f}{27}\right). \quad (4.21)$$

A similar argument applied to $\langle TTR\rangle$ can be used to determine the flow of $c - a$. One finds

$$c_{IR} - a_{IR} = \frac{1}{16}(N_c^2 + 1)$$
$$c_{UV} - a_{UV} = -\frac{1}{16}\left(N_c^2 - 1 - \frac{2}{3}N_f N_c\right) \quad (4.22)$$

(The triangle graph contributions in (4.20-4.22) are normalized consistent with (4.4).) The flows of $5a - 3c$ and $c - a$ agree with the infrared limits of (4.12) and (4.9), respectively.

To conclude, the IR values of the two gravitational central charges in the electric theory are

$$c_{IR} = \frac{1}{16}\left(7\,N_c^2 - 2 - 9\frac{N_c^4}{N_f^2}\right), \quad a_{IR} = \frac{3}{16}\left(2\,N_c^2 - 1 - 3\frac{N_c^4}{N_f^2}\right). \tag{4.23}$$

These results will be discussed in the next section.

5 Discussion

The techniques developed here are quite generally applicable to any supersymmetric gauge theory, provided the theory flows to an IR fixed point and the gauge β-function has the general structure of the NSVZ formula, so that anomalous dimensions can be determined at the fixed point. Specific applications have been made to the electric N=1 $SU(N_c)$ series in the conformal window $3N_c/2 < N_f < 3N_c$ and their magnetic duals.

It should be noted that our formulas for central charge flows do not give new tests of duality because the S_μ current whose anomalies agree in the electric and magnetic theories [2] coincides at the IR fixed point with the R_μ current for which the anomalies are various linear combinations of the IR central charges. We derive here the flows $c_{IR} - c_{UV}$ and $a_{IR} - a_{UV}$ in the magnetic case by this shortcut. The IR values are just those given in (4.23) for the electric theory, and the UV values are determined by the free-field content. In this way we find, in the magnetic theory,

$$c_{IR} - c_{UV} = \frac{1}{24}\left(1 - \frac{3}{2}\frac{N_c}{N_f}\right)\left(9\frac{N_c^3}{N_f} - 6N_c^2 + 6N_f^2 + N_cN_f\right),$$

$$a_{IR} - a_{UV} = -\frac{1}{12}\left(1 - \frac{3}{2}\frac{N_c}{N_f}\right)^2\left(3N_c^2 + 4N_cN_f + 3N_f^3\right). \tag{5.1}$$

In these formulas, γ_q^{IR} is an overall factor and, again, the correct two-loop results in the weakly coupled region of the magnetic theory, $\gamma_q^{IR} \ll 1$, are reproduced. Note once again the appearance of $(\gamma_q^{IR})^2$ in $a_{IR} - a_{UV}$.

Let us discuss our results for central charge flows from the viewpoint of c-theorem expectations. The results (2.28) and (2.36) for central charges of flavor currents in the electric and magnetic theories both satisfy $b_{IR} - b_{UV} > 0$ in the entire conformal window, so there is no c-theorem for flavor.

For the central charge c of the TT OPE and the Euler anomaly coefficient a, there is earlier work by Bastianelli [22] for $SU(N_c)$ SQCD with $N_f = 0$ and in the confinement range $N_c \leq N_f \leq 3N_c/2$. The IR limit is a free theory of massless excitations of independent gauge invariant composite operators. In an empirical approach, Bastianelli computed the free field values in the IR and UV and found both $c_{IR} - c_{UV} < 0$ and $a_{IR} - a_{UV} < 0$, thus suggesting that a c-theorem holds in supersymmetry for the central charges c and a. Our techniques permit

the extension of these c-theorem tests to the conformal window where there is an interacting IR fixed point.

The results for c show both positive and negative flows. In the electric theory $c_{IR} - c_{UV}$ is negative near the lower edge of the conformal window, but positive near the upper edge. In the magnetic theory $c_{IR} - c_{UV} > 0$ in the entire range. We conclude that there is no supersymmetric c-theorem for c. Discussions about this issue have already appeared in the literature in the domain of weakly coupled theories, both supersymmetric , for example in [7] using formula (4.16), and nonsupersymmetric [23].

The story of the Euler central charge a is very different, since we find $a_{IR} - a_{UV} < 0$ for both electric and magnetic theories in the full range of the conformal window. We believe that this is strong (because nonperturbative) new evidence in support of an a-theorem (i.e. a c-theorem for the Euler anomaly) and thus of irreversibility of the RG flow in quantum field theory. This is the first time that these ideas have been tested in strongly coupled theories.

One can also consider the central charge flow when quarks masses are introduced or the Higgs mechanism is used to give masses to some gauge fields and their superpartners. In either case, the IR values of the central charges change, while the UV values are the same as before. One finds, from (4.23) and the free field UV values,

$$a_{IR} - a_{UV} = \frac{1}{48} \left(18N_c^2 - 27\frac{N_c^4}{N_f^2} - 9N_c'^2 - 2N_c'N_f' \right).$$
(5.2)

Here the IR values N_c and N_f are assumed to be in the conformal window, in order to have an IR fixed point, while the UV values N_c' and N_f' are subject only to the conditions $N_c' \geq N_c$, $N_f' \geq N_f$ and $N_f' \leq 3N_c'$. The condition $a_{IR} - a_{UV} < 0$ is satisfied in this more general situation, where some fields are integrated out in the flow.

Let us discuss other positivity conditions suggested by the simple physical intuition that the central charges count degrees of freedom, variously weighted according to the external field one is using (gravitational, flavor, etc.). In particular, one expects the following inequalities to hold,

$$c_{IR} \geq 0, \quad a_{IR} \geq 0, \quad \frac{\partial c_{IR}}{\partial N_f} \geq 0, \quad \frac{\partial a_{IR}}{\partial N_f} \geq 0, \quad \frac{\partial c_{IR}}{\partial N_c} \geq 0, \quad \frac{\partial a_{IR}}{\partial N_c} \geq 0.$$
(5.3)

The first inequality is rigorous, since c appears in the $<TT>$-correlator. The second inequality states that a also should count degrees of freedom and has been recently proposed in [24]. The other inequalities express the expectation that adding matter fields (by varying N_f) or enlarging the color group (by varying N_c) should increase the number of degrees of freedom.

The inequalities (5.3) obviously hold in the UV and we would like to discuss them in the IR. Imposing them on our results (4.23) we find the following restrictions, in the large N_c and N_f limits,

$$N_f^2 \geq \frac{9}{7}N_c^2, \quad N_f^2 \geq \frac{3}{2}N_c^2, \quad \text{n.r.}, \quad N_f^2 \geq \frac{18}{7}N_c^2, \quad \text{n.r}, \quad N_f^2 \geq 3N_c^2,$$
(5.4)

where n.r. means "no restriction". The first inequality, as we said, is rigorous and so puts a limit on the region where the IR fixed point exists. This region contains the full conformal window, in agreement with electric-magnetic duality. We see that below a certain value of N_f our treatment necessarily breaks down, which means that no IR fixed point exists. This is expected, since, for example, pure supersymmetric Yang-Mills theory ($N_f = 0$) has no IR fixed point, according to the NSVZ exact β-function.

The other inequalities of (5.3) have not been proved and so our discussion is purely speculative. One can observe that all but the last one are satisfied in the entire conformal window. So, if one assumes electric-magnetic duality, then the physical intuition that we suggested is not completely correct and there is a region of the conformal window where enlarging the gauge group decreases the value of a in the IR. We do not have a way to resolve this puzzle at the moment, but we believe that these remarks are relevant in order to understand the nature of central charges better.

A final comment concerns the relation between central charges and 't Hooft anomalies. The 't Hooft anomalies are quantities that are constant at all energies, so computable at the free fixed point. They can be regarded as the invariants (or "indices") of a quantum field theory. The central charges, on the other hand, have a direct physical meaning and their RG flow is nontrivial. In supersymmetric theories the RG interpolation problem can be solved because at the IR fixed point the central charges are related to the 't Hooft anomalies. We now show that all the known 't Hooft anomalies can be expressed in terms of the central charges.

The 't Hooft anomalies that contain at least one vertex $U(1)_S$ are related by supersymmetry to appropriate terms in the trace anomaly and therefore to primary central charges. We know that the $U(1)_S$ anomaly is proportional to $c - a$, the $U(1)_S^3$ anomaly is proportional to $5a - 3c$ while the $U(1)_S GG'$ anomaly is related to the flavor central charge b, G and G' denoting flavor groups other than $U(1)_S$. Other $G_1 G_2 G$ 't Hooft anomalies do not contain the vertex $U(1)_S$. They are related to certain secondary central charges c'. Using the construction of Section 3 of [8], let us consider the G-channel of the $\langle G_1(x) G_2(y) G_3(z) G_4(z) \rangle$ four-point function, where the limit $|x - y|, |z - w| \ll |x - z|$ is taken. These secondary central charges are rather special, since the intermediate channel is again a conserved current. The value of c' is simply expressed in terms of a primary central function $b_{G,G}$ and two 't Hooft anomalies. More precisely,

$$c'_{G_i;G} = (G_1 G_2 G) \, b_{G,G} \, (G, G_3, G_4) \tag{5.5}$$

$(G_i G_j G)$ denoting the value of the appropriate triangle anomaly. Therefore the set of primary and (special) secondary central charges contains the set of 't Hooft anomalies. An implication of this remark is that the central charges in question coincide in both the electric and magnetic theories. The secondary central charges that are not included in this set, on the other hand, do not correspond to known 't Hooft anomalies, but they should also coincide in the electric and the magnetic theory. One possible extension of the analysis of this paper is the study of these new central charges, with the purpose of testing non-abelian electric-magnetic duality.

It would be interesting to extend our investigation to nonsupersymmetric theories. The properties of supersymmetry have been intrinsically used in our derivation. In particular,

crucial roles were played by the exact expressions for β-functions and the relation between the trace anomaly and the chiral anomaly of the R-current. In view of this, the nonsupersymmetric generalization is nontrivial. A relatively simple case to start with is pure Yang-Mills theory. The central functions $c(\alpha)$ and $a(\alpha)$ should flow to zero in the IR, since the theory is expected to have a mass gap.

Finally, the other interesting open problem is to rigorously derive the a-theorem (at least in the class of supersymmetric theories) and definitively prove that the RG flow is irreversible in quantum field theory.

6   Acknowledgements

A   Appendix

We present in this Appendix a superspace derivation of the coefficients $c - a$ and $5a - 3c$ appearing in (4.7).

For any matter system coupled to (background) supergravity the supercurrent and supertrace are defined by

$$J_{\alpha\dot\alpha} = \frac{\delta\Gamma}{\delta H^{\alpha\dot\alpha}} \quad , \quad J = \frac{\delta\Gamma}{\delta\phi^3} \tag{A.1}$$

where Γ is the (classical or quantum) action, $H^{\alpha\dot\alpha}$ is the supergravity vector prepotential superfield, and ϕ is the superconformal (chiral superfield) compensator. The superfield $H^{\alpha\dot\alpha}$ contains the vierbein and, as its last, $\theta^2\bar\theta^2$ component, the axial vector auxiliary field denoted here by $V^{\alpha\dot\alpha}$. The supercurrent and supertrace are related by the (local supersymmetry) conservation equation

$$\nabla^{\dot\alpha} J_{\alpha\dot\alpha} = \nabla_\alpha J \tag{A.2}$$

If the theory is superconformal the action is independent of the compensator and the supertrace vanishes. Otherwise, J contains the trace anomaly, the supersymmetry current γ-trace anomaly, and the chiral anomaly for the R-current. The R-current appears as the first component of the supercurrent or, equivalently, as the term in the action that couples to $V^{\alpha\dot\alpha}$.

For supersymmetric matter systems (scalar or vector multiplets) the supertrace has been computed in terms of so-called super-b_4 coefficients in refs. [25, 26] (however, we use in this

Appendix the conventions of *Superspace* [27]). One finds

$$J = \frac{1}{24\pi^2}[cW^2 - a\Xi_c] \tag{A.3}$$

where the numerical coefficients c, a take the values

$$\text{Scalar multiplet}: \quad c = \frac{1}{24} \quad , \quad a = \frac{1}{48}$$
$$\text{Vector multiplet}: \quad c = \frac{3}{24} \quad , \quad a = \frac{9}{48} \tag{A.4}$$

Here $W^2 = \frac{1}{2}W_{\alpha\beta\gamma}W^{\alpha\beta\gamma}$ is the square of the superWeyl tensor, while

$$\Xi_c = W^2 + (\bar{\nabla}^2 + R)(G^2 + 2\bar{R}R) \tag{A.5}$$

is the chirally projected superEuler density. More precisely

$$\mathcal{E} = \frac{1}{(4\pi)^2}\left[\int d^4x d^2\theta \phi^3 \Xi_c + \int d^4x d^2\bar{\theta}\bar{\phi}^3\bar{\Xi}_c\right] \tag{A.6}$$

$$\mathcal{P} = \frac{1}{(4\pi)^2}\left[\int d^4x d^2\theta \phi^3 \Xi_c - \int d^4x d^2\bar{\theta}\bar{\phi}^3\bar{\Xi}_c\right] = \frac{1}{(4\pi)^2}\left[\int d^4x d^2\theta \phi^3 W^2 - \int d^4x d^2\bar{\theta}\bar{\phi}^3\bar{W}^2\right]$$

give the Euler number and Pontrjagin number, respectively. The chiral superfields $W_{\alpha\beta\gamma}$ and R, and the real superfield $G_{\alpha\dot\alpha}$ are the three superspace curvatures. In our conventions $G^2 = \frac{1}{2}G^{\alpha\dot\alpha}G_{\alpha\dot\alpha}$.

We are interested in the component trace and R-current anomalies for the matter system in a background gravity and axial-vector auxiliary field R-current source. The trace of the stress-tensor is obtained from the $\theta^\alpha\bar{\theta}^{\dot\alpha}$ component of the supercurrent:

$$\Theta = \frac{3}{8}[\nabla^\alpha, \bar{\nabla}^{\dot\alpha}]J_{\alpha\dot\alpha}|_{\theta=0} = \frac{3}{4}(\nabla^2 J + \bar{\nabla}^2\bar{J})|_{\theta=0} \tag{A.7}$$

while the divergence of the R-current is obtained from the first component of the supercurrent:

$$i\nabla^a R_a = \frac{1}{2}\{\nabla^\alpha, \bar{\nabla}^{\dot\alpha}\}J_{\alpha\dot\alpha}|_{\theta=0} = (\nabla^2 J - \bar{\nabla}^2\bar{J})|_{\theta=0} \tag{A.8}$$

One can obtain the corresponding component anomalies by brute force θ-expansion or, as we shall do here, by defining components by projection and exploiting the Bianchi identities of the theory. In doing so, we will obtain component results expressed in terms of the component curvature tensor R_{abcd} and the axial vector field strength, V_{ab}. However, one further step is necessary: the standard supergravity constraints [27] lead to component covariant derivatives with nonzero torsion, due to both the gravitino fields (that we set to zero here), and to the axial vector auxiliary field. Before reading off the component anomalies, we must separate the curvature in terms of the ordinary Einstein curvature and additional terms, again proportional to the field strength V_{ab}.

We obtain the component results as follows: the W^2 term in the supertrace leads to

$$\frac{1}{2}\nabla^2 W^2 \pm \frac{1}{2}\bar{\nabla}^2\bar{W}^2 = \frac{1}{4}\nabla^\delta W^{\alpha\beta\gamma}\nabla_\delta W_{\alpha\beta\gamma} - \frac{1}{4}W_{\alpha\beta\gamma}\nabla^2 W^{\alpha\beta\gamma} \pm h.c. \tag{A.9}$$

evaluated at $\theta = 0$, the $\pm$ sign corresponding to either the trace or the chiral anomaly. Since we are restricting ourselves to a bosonic background the second term (proportional at $\theta = 0$ to the gravitino field strength) may be dropped. We write each factor in the first term as the sum of symmetrized and antisymmetrized (in δ and the already symmetrized $\alpha\beta\gamma$) indices, i.e.

$$\nabla_\delta W_{\alpha\beta\gamma} = \frac{1}{6}[\nabla_{(\delta}W_{\alpha\beta\gamma)} + C_{\alpha\delta}\nabla^\lambda W_{\lambda\beta\gamma} + C_{\beta\delta}\nabla^\lambda W_{\lambda\gamma\alpha} + C_{\gamma\delta}\nabla^\lambda W_{\lambda\alpha\beta}] \tag{A.10}$$

For the last three terms we use the Bianchi identity $\nabla^\lambda W_{\lambda\beta\gamma} = \frac{i}{2}\nabla_{(\beta}^{\ \dot\beta}G_{\gamma)\dot\beta}$. Evaluating at $\theta = 0$, the (totally symmetrized) first term gives the self-dual part (in spinor notation) of the component Weyl tensor, $W_{\alpha\beta\gamma\delta} = \frac{1}{4!}\nabla_{(\delta}W_{\alpha\beta\gamma)}$ (but we emphasize again that it still contains some torsion pieces). The second term, using the definition of the axial vector auxiliary field $V_a = G_a|_{\theta=0}$ leads to the self-dual part of its field strength, $V_{\beta\gamma} = \frac{1}{2}\partial_{(\beta\dot\beta}V_{\gamma)}^{\ \dot\beta}$.

For the G^2 terms in the superEuler density, dropping irrelevant terms, we have

$$(\nabla^2\bar\nabla^2 \pm \bar\nabla^2\nabla^2)G^2 = -\frac{1}{2}[\nabla^\alpha\bar\nabla^{\dot\alpha}G^{\beta\dot\beta}\nabla_\alpha\bar\nabla_{\dot\alpha}G_{\beta\dot\beta} \pm \bar\nabla^{\dot\alpha}\nabla^\alpha G^{\beta\dot\beta}\bar\nabla_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta}] \tag{A.11}$$

With suitable (anti)symmetrization and use of the Bianchi identity $\nabla^{\dot\alpha}G_{\alpha\dot\alpha} = \nabla_\alpha R$, one has, for example,

$$\bar\nabla_{\dot\alpha}\nabla_\alpha G_{\beta\dot\beta} = \frac{1}{4}\bar\nabla_{(\dot\alpha}\nabla_{(\alpha}G_{\beta)\dot\beta)} + \frac{1}{4}C_{\dot\beta\dot\alpha}\nabla_{(\alpha}^{\ \dot\gamma}G_{\beta)\dot\gamma} + \frac{1}{2}C_{\dot\beta\dot\alpha}C_{\beta\alpha}\bar\nabla^2\bar R \tag{A.12}$$

At $\theta = 0$ the middle term is again expressible in terms of the selfdual part of V_{ab}. The first and last terms can be shown to be expressible in terms of the (torsionful) Ricci tensor and scalar. as follows: we have the identity expressing the component curvature tensor $R_{abcd} = C_{\dot\gamma\dot\delta}R_{ab\gamma\delta} + C_{\gamma\delta}R_{ab\dot\gamma\dot\delta}$ in terms of superspace objects [27, 28]

$$R_{ab\gamma\delta} \equiv R_{\alpha\dot\alpha,\beta\dot\beta,\gamma\delta} = \frac{1}{6}C_{\dot\alpha\dot\beta}\nabla_{(\alpha}W_{\beta\gamma\delta)} - \frac{1}{4}C_{\alpha\beta}\nabla_{(\dot\alpha}\nabla_{(\gamma}G_{\delta)\dot\beta)}$$
$$- \frac{i}{12}C_{\dot\alpha\dot\beta}[C_{\beta\gamma}\nabla_{(\alpha}^{\ \dot\gamma}G_{\delta)\dot\gamma} + C_{\beta\delta}\nabla_{(\alpha}^{\ \dot\gamma}G_{\gamma)\dot\gamma}] + \frac{1}{2}C_{\dot\alpha\dot\beta}C_{\beta(\gamma}C_{\delta)\alpha}(\bar\nabla^2\bar R + 2R\bar R) \tag{A.13}$$

where the right hand side is evaluated at $\theta = 0$. In particular

$$\bar\nabla_{(\dot\alpha}\nabla_{(\gamma}G_{\delta)\dot\beta)}|_{\theta=0} = 2R^\beta_{\ (\dot\alpha,\beta\dot\beta),\gamma\delta} \tag{A.14}$$

and, similarly

$$\bar\nabla^2\bar R + 2R\bar R|_{\theta=0} = -\frac{1}{6}R^{\gamma\dot\alpha\ \delta}_{\ \ ,\dot\alpha,\gamma\delta} \tag{A.15}$$

Our final ingredient expresses the component torsion and the torsionful connection in terms of the axial vector auxiliary field and the torsionless connection [27, 28]:

$$T_{ab}^{\ \ c} \equiv T_{\alpha\dot\alpha,\beta\dot\beta}^{\ \ \ \ \gamma\dot\gamma} = i\left(C_{\alpha\beta}\delta_{\dot\alpha}^{\ \dot\gamma}V_{\dot\beta}^{\ \gamma} - C_{\dot\alpha\dot\beta}\delta_\alpha^{\ \gamma}V_\beta^{\ \dot\gamma}\right)$$
$$\omega_{abc} = \omega_{abc}(e) - \frac{1}{2}\varepsilon_{abcd}V^d \tag{A.16}$$

This allows rewriting the component torsionful curvatures in terms of the ordinary curvature tensor and additional V_{ab} terms.

Assembling all the ingredients, we finally obtain the following contributions to the trace and R-current anomalies:

$$\Theta = \frac{c}{32\pi^2}[(W_{\alpha\beta\gamma\delta})^2 + (\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}})^2] + \frac{c}{3\pi^2}[(V_{\alpha\beta})^2 + (\bar{V}_{\dot{\alpha}\dot{\beta}})^2]$$
$$- \frac{a}{32\pi^2}[(W_{\alpha\beta\gamma\delta})^2 + (\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}})^2 - 2R^{\alpha\beta\dot{\alpha}\dot{\beta}}R_{\alpha\beta\dot{\alpha}\dot{\beta}} + \frac{8}{3}(R_{\alpha\beta}{}^{\alpha\beta})^2]$$
$$\partial^a R_a = \frac{c-a}{24\pi^2}[(W_{\alpha\beta\gamma\delta})^2 - (\bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}})^2] - 4\frac{5a-3c}{9\pi^2}[(V_{\alpha\beta})^2 - (\bar{V}_{\dot{\alpha}\dot{\beta}})^2] \qquad (A.17)$$

in terms of selfdual and anti-selfdual parts of the curvature tensors and the axial vector field strength. This can be rewritten in the form given in (4.3) and (4.7).

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